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Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Extending the Jump Analysis for Aerodynamic Asymmetry

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Nomenclature

a_v = integration constant, $\rho C_{x0} DS/2m$
C_i = projectile aerodynamic coefficients
D = projectile characteristic length (diameter)
$\bar{F}$ = body force component caused by asymmetry
G = scaled gravitational constant, gD/V_0
g = gravitational constant
I_X, I_Y = mass moments of inertia
J_I, K_I = inertial frame transverse Cartesian unit vectors
K_p = constant defined as $-\rho DS(2C_{X0}I_x + mC_{LP}D^2)/4mI_x$
$\bar{M}$ = body moment component caused by asymmetry
m = projectile mass
p, $\tilde{q}$, $\tilde{r}$ = angular velocity components vector of projectile in the no-roll frame
S = surface area, $\pi D^2/4$
s = nondimensional arc length
u, $\tilde{v}$, $\tilde{w}$ = mass center velocity components in the no-roll reference frame
V_0 = forward velocity of projectile
$\{x \;\; y \;\; z\}$ = position vector of body center of mass in an inertial reference frame
Z = substitution variable defined as ϕ'_∞/K_p
Γ = substitution variable defined as $1 - \phi'_0/\phi'_\infty$
Δ_J, Δ_K = $\boldsymbol{J_I}$ and $\boldsymbol{K_I}$ components of aerodynamic jump
Π = complex aerodynamic jump caused by asymmetry
$\boldsymbol{\Pi}$ = jump vector in inertial frame caused by body asymmetry
ρ = air density
ϕ = Euler roll angle of the projectile

Introduction

THIS study extends the work of Murphy and Bradley[1] and Fansler and Schmidt[2] describing jump of a varying roll rate projectile caused by a slight configuration asymmetry. Their analysis uses projectile linear theory, and the jump caused by asymmetry was expressed as an integral. They evaluated the jump integral using small parameter power series, large parameter asymptotic series, and numerical integration. An equivalent integral arrived herein is shown to be expressed in terms of a confluent hypergeometric function that is calculated accurately without the need of the previous size restraints on the jump parameters. Using asymmetry as a possible way to guide a free-flight projectile requires fast onboard calculations of the jump integral. Algorithms for any of the preceding evaluation methods are generally too slow. To address this problem, a continued fraction is proposed to approximate the confluent hypergeometric function and is shown to give an accurate rational expression for asymmetric jump, resulting in fast calculations.

Linear Theory

A typical representation of linear theory[1] for projectile flight, is given by Cooper and Costello.[3] They give results of a side thruster exerting a finite-duration impulse on a projectile during free flight. The same linear theory is used here, along with the identical nomenclature, where the side impulse is now replaced with terms representing forces and moments caused by a configuration asymmetry. A particular example could be a finned projectile (Fig. 1), with one fin bent or damaged, thus causing an asymmetry. In linear theory, the lateral, translational, and rotational velocity components are transformed to a nonrolling reference frame.[3] The nonrolling frame, or so-called fixed plane, proceeds with only precessional and nutational rotations from an inertial reference frame. Transforming to the fixed plane puts Cooper and Costello[3] equations of motion in the following form:

$$x' = D \tag{1}$$

$$y' = \frac{D}{V}\tilde{v} + D\psi \tag{2}$$

$$z' = \frac{D}{V}\tilde{w} - D\theta \tag{3}$$

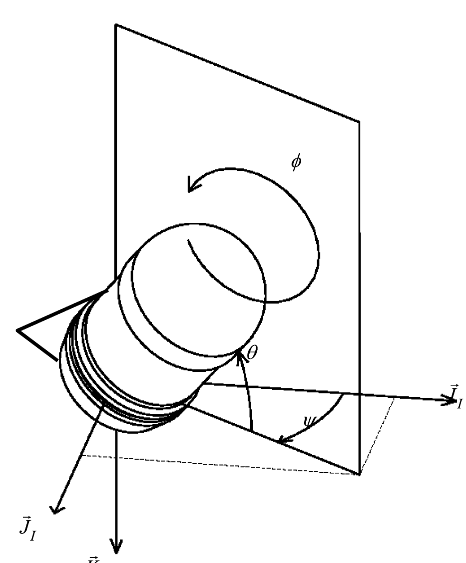

Fig. 1 Projectile orientation definition with asymmetry.

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$$\phi' = \frac{D}{V} p \tag{4}$$

$$\theta' = \frac{D}{V} \tilde{q} \tag{5}$$

$$\psi' = \frac{D}{V} \tilde{r} \tag{6}$$

$$V' = -\frac{\rho S D C_{X0}}{2m} V - \frac{D g \theta}{V} \tag{7}$$

$$p' = \frac{\rho S D^2 C_{LDD}}{2 I_X} V + \frac{\rho S D^3 C_{LP}}{4 I_X} p \tag{8}$$

$$\begin{Bmatrix} \tilde{v}' \\ \tilde{w}' \\ \tilde{q}' \\ \tilde{r}' \end{Bmatrix} = \begin{bmatrix} -A & 0 & 0 & -D \\ 0 & -A & D & 0 \\ B/D & C/D & E & -F \\ -C/D & B/D & F & E \end{bmatrix} \begin{Bmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{q} \\ \tilde{r} \end{Bmatrix} + \begin{Bmatrix} \bar{F} V \cos(\phi) \\ \bar{F} V \sin(\phi) + G \\ -\bar{M} V \sin(\phi)/D \\ \bar{M} V \cos(\phi)/D \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} A \\ B \\ C \\ E \\ F \end{Bmatrix} = \begin{Bmatrix} \dfrac{\pi \rho D^3 C_{NA}}{8m} \\ \dfrac{\pi \rho p D^5 C_{YPA} (SL_{MAG} - SL_{CG})}{16 I_y V} \\ \dfrac{\pi \rho D^4 C_{NA} (SL_{COP} - SL_{CG})}{8 I_y} \\ \dfrac{\pi \rho D^5 C_{MQ}}{16 I_y} \\ \dfrac{p D I_x}{I_y V} \end{Bmatrix} \tag{10}$$

Without loss of generality the angular position of the asymmetry is taken to be at roll angle $\phi = 0$.

The Euler roll rates are

$$\begin{bmatrix} \phi'_0 \\ \phi'_\infty \end{bmatrix} = \begin{bmatrix} \phi'(s=0) \\ \rho C_{LDD} D^3 S (2 I_x K_p)^{-1} \end{bmatrix}$$

The last vector on the right-hand side of Eq. (9) contains the body force $\bar{F}$ and body moment $\bar{M}$, components of the assumed asymmetry. Multiplication by $\cos(\phi)$ and $\sin(\phi)$ rotates the body forces and moments into the fixed plane. Aside from the fact that V appears in some of the coefficients, the dynamics are now expressed with linear ordinary differential equations. Note that variables with tildes represent fixed plane variables and a prime represents a derivative with respect to the nondimensional arc length s given by

$$s = \frac{1}{D} \int_0^t V \, \mathrm{d}t \tag{11}$$

Linear Theory Solution

Because V changes slowly with respect to the other variables, it is considered constant, $V \approx V_0$, when it appears as a coefficient in all dynamic equations, except its own, and the roll equation. The solution to the differential equation (7), for the forward velocity when taking $\theta_0 \ll 1$ (Ref. 3) is

$$V(s) = V_0 e^{-a_V s} \tag{12}$$

Noting that $p(s) = \phi' V / D$ and using Eq. (12) transforms the roll equation (8) to

$$\frac{\mathrm{d}^2 \phi}{\mathrm{d} s^2} + K_p \frac{\mathrm{d} \phi}{\mathrm{d} s} = \frac{\rho C_{LDD} D^3 S}{2 I_x} \tag{13}$$

When solved and with the assumption that the roll angle starts at $\phi_0 = 0$, that is, the angular position of the asymmetry, implies

$$\phi = (\phi'_\infty - \phi'_0)(e^{-K_p s} - 1) \big/ K_p + \phi'_\infty s \tag{14}$$

The product of damping and the product AE are taken to be vanishingly small because the density ratio is assumed small[4]; see Eq. (10). Defining $\xi \equiv (\tilde{v} + i \tilde{w}) / V_0$, and after some manipulations, enables Eq. (9) to be reduced to the following single differential equation[4]:

$$\xi'' - (E - A)\xi' - (AE + C)\xi = -(\bar{M} + E\bar{F}) e^{i(\phi + \phi_B)} - i E G / V_0 \tag{15}$$

where it is assumed p is relatively small enough to be ignored. Rather than solve Eq. (15), a more direct way to obtain the effects of jump caused by asymmetry is to solve the swerve differential equations that govern the center of mass motion.

Swerve

The lateral, translation, and rotational, velocity components are contained in the attitude differential equations, and the attitudes are contained within the swerve differential equations. Swerving motion along the Earth-fixed $\boldsymbol{J}_I$ and $\boldsymbol{K}_I$ axes results from a combination of the normal aerodynamic forces, as the projectile pitches and yaws, plus the forces and moments due to the configuration asymmetry. Differentiating Eqs. (2) and (3) with respect to s and using the definition of ξ with the aid of Eq. (15) leads to the following expression for swerve[1]:

$$(C/D)(y'' + i z'') = -(A\bar{M} - C\bar{F}) e^{i\phi} + (i C G / V_0) + (E - A) A \xi' - A \xi'' \tag{16}$$

For a stable projectile, the swerve caused by epicyclical vibration decays as the projectile progresses downrange and does not affect the long-term lateral motion.[1] However, other contributions including the asymmetry cause integrated effects that contribute to the long-term lateral motion of the projectile. This theory shows the center of mass motion contains terms that are bounded with arc length s, and with the inclusion of gravity, the solution will have higher-order diverging terms. These diverging terms are typically denoted as gravity drop. The linear terms are called aerodynamic jump terms, which are caused by initial conditions at the gun muzzle and the projectile's aerodynamic characteristics. The remaining bounded terms result from the configuration asymmetry. Ignoring gravity and evaluating the following limits formally defines the total aerodynamic jump as

$$\lim_{s \to \infty} \{[y(s)/D]/s\} = \Delta_J, \qquad \lim_{s \to \infty} \{[z(s)/D]/s\} = \Delta_K \tag{17}$$

The total jump vector is expressed as the sum of two vectors. The first vector represents the muzzle conditions; these come from the linear s terms, and will not be discussed here. The interested reader is referred to the literature[3,4] for a complete description of these contributions. With focus on the second vector, $\boldsymbol{\Pi}$, because of asymmetry on the projectile that is subjected to a varying roll rate,

$$\Pi = -J_A \Phi, \qquad J_A = -\frac{A\bar{M} - C\bar{F}}{C}$$

$$\Phi = \lim_{s \to \infty} \frac{1}{s} \int_0^s \int_0^\tau e^{i\phi(\sigma)} \, \mathrm{d}\sigma \, \mathrm{d}\tau \tag{18}$$

Here the complex valued asymmetric jump Π is interpreted such that the real part is in the direction $\boldsymbol{J}_I$ and the imaginary part is directed parallel to $\boldsymbol{K}_I$. The function Φ is derived in the Appendix and has the form

$$\Phi = \{(i/Z) + [\Gamma/(1 - iZ)] F_1(1; 2 - iZ - i\Gamma Z)\} \tag{19}$$

where $\Gamma = 1 - \phi'_0/\phi'_\infty$, $Z = \phi'_\infty / K_p$ and ${}_1F_1$ is the confluent hypergeometric function.[5] Constant rolling motion $\Gamma = 0$ implies

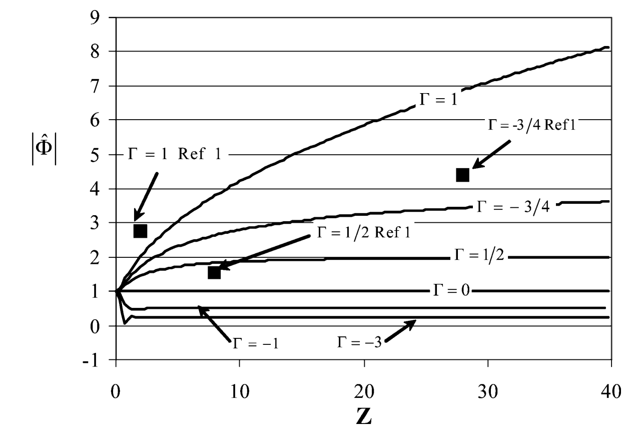

Fig. 2 Magnitude of $\hat{\Phi}$ as function of *Z*.

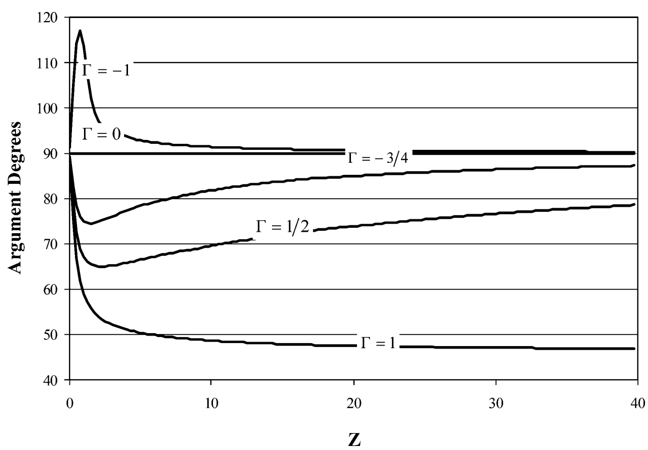

Fig. 3 Argument of $\hat{\Phi}$ as function of *Z*.

$\Phi = iZ^{-1}$, which means the jump is orientated 90 deg to the initial angular position of the asymmetry. Using this value as the standard to compare jumps when spin is varying suggests rescaling Φ so that the function of interest is now $\hat{\Phi} = \Phi Z$, so that

$$\hat{\Phi} = i + \Gamma Z(1 - iZ)^{-1}{}_1F_1(1; 2 - Zi, -i\Gamma Z) \tag{20}$$

Polar plots of Eq. (20) for discrete values of Γ and a continuous variation of Z are given in Fig. 2 for $|\hat{\Phi}|$, and Fig. 3 has the argument of $\hat{\Phi}$. Apparently neither Murphy and Bradley[1] nor Fansler and Schmidt[2] were aware of Eq. (20), which makes their small parameter and asymptotic analysis unnecessary to calculate asymmetric jump. Software is now readily available to calculate ${}_1F_1(1 : 2 - Zi, -i\Gamma Z)$ (Ref. 6).

Faster Calculations

Deliberate asymmetries could be considered as a method to influence projectile jump as a means of guidance. This possibility gives motivation for finding a faster way to use Eq. (20) when onboard calculations are necessary.

Successive integration by parts of the last expression shows

$$-i\hat{\Phi} = 1 - \frac{\Gamma}{\Gamma - 1} + \frac{i\Gamma}{Z(\Gamma - 1)^3} - \frac{\Gamma(2\Gamma + 1)}{Z^2(\Gamma - 1)^5} + \frac{i\Gamma}{Z^2}$$
$$\int_0^1 F_4' \exp\{-iZ[\Gamma y - \ell n(1 - y)]\}\,\mathrm{d}y \tag{21}$$

for which

$$F_0 = \frac{i(y - 1)}{Z[\Gamma(y - 1) + 1]}, \qquad F_n = F_0 F_{n-1}'$$

This is an asymptotic series expansion for large $Z \gg 1$, and inspection shows that as $Z \to \infty$, that is, $K_p \to 0$,

$$\hat{\Phi} \to i/(1 - \Gamma) = i(\phi_\infty'/\phi_0') \tag{22}$$

which says that the jump has magnitude ϕ_∞'/ϕ_0' and is oriented at a right angle to the initial asymmetry orientation. Equation (20) can also be written as a Kummer series (see Ref. 5) giving the following expansion:

$$-i\hat{\Phi} = 1 - \frac{iZ\Gamma}{1 - iZ} - \frac{Z^2\Gamma^2}{(1 - iZ)(2 - iZ)}$$
$$+ \frac{iZ^3\Gamma^3}{(1 - iZ)(2 - iZ)(3 - iZ)}$$
$$- \frac{Z^4\Gamma^4}{(1 - iZ)(2 - iZ)(3 - iZ)(4 - iZ)} - \cdots \tag{23}$$

and clearly the limit of Eq. (23) as $Z \to \infty$ is

$$-i\hat{\Phi} = 1 + \Gamma + \Gamma^2 + \Gamma^3 + \Gamma^4 + \cdots = 1/(1 - \Gamma) \tag{24}$$

for $\Gamma < 1$. This result, in light of the limit of the asymptotic series Eq. (22), suggests the Kummer series when transformed to a continued fraction may produce an accurate approximation to $\hat{\Phi}$ for a wide range of Γ values. To continue the investigation, the assumption was made that a reasonable approximation has the following representation:

$$-i\hat{\Phi} - 1 = a0\Gamma/\big(1 + a1\Gamma/\big(1 + a2\Gamma/\big(1 + a3\Gamma/$$
$$\cdots\big(1 + a7\Gamma\big)\big)\big)\big)\big)\big)\big) \tag{25}$$

whereupon finding the coefficients $a0, a1, a2, \ldots, a7$ results in

$$a0 = \frac{Z}{Z + i}, \qquad a1 = \frac{-Z}{Z + 2i}, \qquad a2 = \frac{iZ}{(Z + 2i)(Z + 3i)}$$
$$a3 = \frac{-Z(Z + 2i)}{(Z + 3i)(Z + 4i)}, \qquad a4 = \frac{2iZ}{(Z + 4i)(Z + 5i)}$$
$$a5 = \frac{-Z(Z + 3i)}{(Z + 5i)(Z + 6i)}, \qquad a6 = \frac{3iZ}{(Z + 6i)(Z + 7i)}$$
$$a7 = \frac{-Z(Z + 4i)}{(Z + 7i)(Z + 8i)} \tag{26}$$

Taking the limit of these expressions for $Z \to \infty$ forces Eq. (26) to yield $\hat{\Phi} \to i/(1 - \Gamma)$, which agrees with Eq. (24). This indicates that Eqs. (25) and (26) form a reasonable rational approximation to Eq. (20). Figure 4 shows some comparisons between the exact solution (20) and this rational approximation construction for various

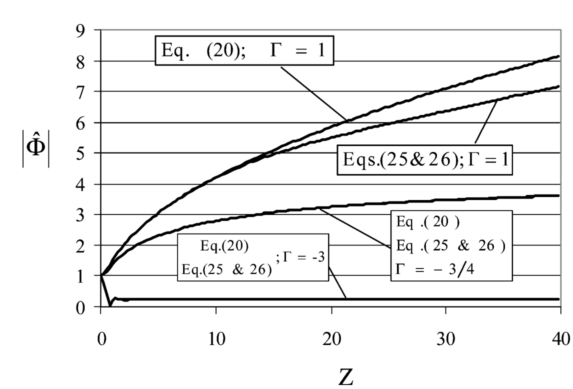

Fig. 4 Comparison of exact vs rational approximation of $\hat{\Phi}$.

values of Γ. Note that, even for the case where $\Gamma = 1$, the comparison shows good agreement for $Z < 20$. Larger values of Z, when $\Gamma = 1$, should be calculated using numerical software for evaluating Eq. (20).

Conclusions

The previous analysis, based on projectile linear theory describing jump due to asymmetry, did not recognize that the jump integral can be written in terms of a confluent hypergeometric function. This result can be calculated using numerical software, but a simpler and faster way to calculate the jump integral is to use the rational expression generated from Eqs. (25) and (26). This expression shows that Eq. (20) is well approximated for a wide range of the pertinent jump parameters. Results presented in this analysis extend the Murphy and Bradley[1] and Fansler and Schmidt[2] results, which could prove useful for fast numerical calculations regarding analysis of asymmetric jump of a free-flight projectile.

Appendix: Integral

Let $\Gamma = 1 - \phi'_0/\phi'_\infty$, $Z = \phi'_\infty/K_p$ and $\tau = sK_p$, then Eq. (14) will be written as

$$\phi(\tau) = Z[\tau + \Gamma(e^{-\tau} - 1)] \tag{A1}$$

Then

$$\Phi = \lim_{s\to\infty} \frac{1}{s} \int_0^s \int_0^\sigma e^{i\phi(\tau)}\,\mathrm{d}\tau\,\mathrm{d}\sigma$$

$$= \lim_{s\to\infty} \frac{1}{s} \int_0^s (s-\tau)\exp^{\{iZ[\tau + \Gamma(e^{-\tau} - 1)]\}}\,\mathrm{d}\tau \tag{A2}$$

Ignoring the limiting process for the moment and writing

$$e^{iZ\Gamma e^{-\tau}} = \sum_{n=0} \frac{(iZ\Gamma)^n e^{-n\tau}}{n!} \tag{A3}$$

makes Eq. (A2) take the form

$$s\Phi = e^{-iZ\Gamma} \sum_{n=0} \frac{(iZ\Gamma)^n}{n!} \int_0^s (s-\tau)e^{(iZ-n)\tau}\,\mathrm{d}\tau \tag{A4}$$

Integrating the last expression and taking the limit $s \to \infty$ causes Eq. (A2) to become

$$\Phi = e^{-iZ\Gamma}\left[\frac{i}{Z} + \sum_{n=1} \frac{(iZ\Gamma)^n}{(n-iZ)n!}\right] \tag{A5}$$

and writing the last summation as the sum of an even plus an odd series gives

$$\Phi = e^{-iZ\Gamma}\left[\begin{array}{c} \dfrac{i}{Z} \\ + \displaystyle\sum_{n=1} \frac{(-1)^n (Z\Gamma)^{2n}}{(2n)!} \int_0^1 t^{(2n-1-iZ)}\,\mathrm{d}t \\ + i\displaystyle\sum_{n=0} \frac{(-1)^n (Z\Gamma)^{2n+1}}{(2n+1)!} \int_0^1 t^{(2n-iZ)}\,\mathrm{d}t \end{array}\right] \tag{A6}$$

The last equation is now written as

$$\Phi = e^{-iZ\Gamma}\left[\begin{array}{c} \dfrac{i}{Z} \\ + \displaystyle\int_0^1 \frac{(\cos(Z\Gamma t) - 1)}{t} t^{-iZ}\,\mathrm{d}t \\ + i\displaystyle\int_0^1 \frac{\sin(Z\Gamma t)}{t} t^{-iZ}\,\mathrm{d}t \end{array}\right] \tag{A7}$$

which after partial integration leads to[5]

$$\Phi = \frac{i}{Z} + \Gamma \int_0^1 e^{iZ\Gamma(t-1)} t^{-iZ}\,\mathrm{d}t = \frac{i}{Z} + \Gamma \int_0^1 e^{-iZ\Gamma y}(1-y)^{-iZ}\,\mathrm{d}y$$

$$= \frac{i}{Z} + \frac{\Gamma}{1-iZ}\,{}_1F_1(1; 2-iZ, iZ\Gamma) \tag{A8}$$

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